An homological analogue of the Baum-Connes conjecture with coefficients for Lie groups

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Abstract

The Baum-Connes conjecture for Lie groups establishes a link between the tempered dual of a real Lie group and the unitary dual of its maximal compact subgroup. This conjecture has been proved in three different ways: via representation-theoretic arguments, using the Dirac-dual Dirac method and more recently via Mackey analogy. Conversely, the Baum-Connes conjecture for Lie groups with coefficients still remains unproven. In this talk we propose an homological analogue of this conjecture with coefficients by comparing the periodic cyclic homology of smooth crossed product algebras. Our proof relies mostly on the adaptation of the Dirac-dual Dirac method to the Fréchet framework.

Motivation

Take a real Lie group G and K a maximal compact subgroup of it. There exists a diffeomorphism between the associated homogeneous space G/K and a contractible euclidian space $\mathbb{R}^q$. The simplicity of this geometry makes the tempered representations of G in relation with the unitary irreducible representations of K.

The link between these spaces of representations can be written using $C^\star$-algebras. Indeed, the tempered dual $\widehat{G}_t$ and the unitary dual $\widehat{K}$ correspond respectively to the topological spaces of primitive ideals of the reduced groups $C^\star$-algebras $C_r^\star(G)$ and $C_r^\star(K)$. The Baum Connes conjecture for Lie groups claims that there exists a degree-shift isomorphism

$$K_\bullet(C_r^\star(K)) \xrightarrow{\sim} K_{\bullet+q}(C_r^\star(G)).$$

There exists three different proofs of this conjecture. In chronological order, the first one is due to A. Wassermann who exhibited a correspondence between the unitary dual of K and the tempered dual of G and established a one-to-one correspondence at the level of representation theory. Then A. Connes and G. Kasparov proposed an approach using bivariant K-theory, and especially the *Dirac-dual Dirac method*, to recover this isomorphism. Finally, N. Higson and A. Afgoustidis gave a geometrical proof known as *Mackey analogy* by deforming smoothly the group G to is Cartan motion group $G_0 = K \times \mathbb{R}^q$ whose K-theoretical invariants are related to the ones of the maximal compact subgroup.

If the real Lie group G acts on a $C^\star$-algebra A, we can construct two reduced crossed products algebras $A \rtimes_r G$ and $A \rtimes_r K$ which can be thought as the reduced group $C^\star$-algebras *with coefficients.* This algebra encodes both representations of the group G and modules over the algebra A. The Baum-Connes conjecture *with coefficients* for Lie groups claims an analogous statement as before, which is the existence of a degree-shift isomorphism

$$K_\bullet(A \rtimes_r K) \longrightarrow K_{\bullet+q}(A \rtimes_r G). \tag{1}$$

This conjecture is still unproved.

In this talk we propose a geometrical counterpart of the Baum-Connes conjecture with coefficients for Lie groups (after stabilization) namely replacing the reduced crossed product algebras by *smooth crossed product algebras* and the K-theoretical invariants by the *periodic cyclic homology groups.*

Setup

Consider a real Lie group G acting smoothly on a locally convex algebra A. We define the **smooth** crossed product of A by G as:

$$A \rtimes G := \mathcal{C}_c^\infty(G, A) \quad (f_1 \star f_2)(g) = \int_G f_1(h) h \cdot f_2(h^{-1}g) dh \in A.$$

Its K-theoretical invariants are replaced by their geometrical counterpart, namely the **periodic cyclic homology**, denoted HP.

In the early 90-s, V. Nistor established a result similar to conjecture (1) for the periodic cyclic homology of smooth crossed products. He constructed an isomorphism [Nis93b]

$$HP_\bullet(A \rtimes K)_{\langle x \rangle} \simeq HP_{\bullet + \dim(G/K)}(A \rtimes G)_{\langle x \rangle}. \tag{2}$$

where the subscript denotes the localization at the maximal ideal of the algebra of class functions on G given by functions vanishing at the conjugacy class $\langle x \rangle$. However, these local isomorphisms cannot be glued together to obtain a global correspondence between the homology groups of $A \rtimes K$ and $A \rtimes G$.

In this talk, we construct a global isomorphism (after stabilization) between these periodic cyclic homology groups, thereby providing a homological analogue of the Baum-Connes conjecture with coefficients for Lie groups. Our approach decomposes this isomorphism into two different pieces via an intermediate space:

$$HP_\bullet(A \rtimes K) \simeq HP_\bullet(\mathcal{C}_c^\infty(G/K, A) \rtimes G) \xrightarrow{\sim} HP_{\bullet + \dim(G/K)}(A \rtimes G). \tag{3}$$

The left-hand isomorphism is well-known at the $C^\star$-algebraic level and requires some adaptations to fit the Fréchet algebras framework, we won't develop it during the talk.

The right-hand isomorphism, way more complicated, follows the following strategy. In the easiest case where $A = \mathbb{C}$ is the trivial algebra, this isomorphism essentially relies on an identification of $HP_\bullet(\mathcal{C}_c^\infty(G/K))$ with $HP_{\bullet + \dim(G/K)}(\mathbb{C})$ in a way that is compatible with crossed products. In K-theory, a Thom isomorphism (Bott periodicity) holds:

$$K_\bullet\left(\mathcal{C}_0(G/K)\right) \xrightarrow{\sim} K_{\bullet + \dim(G/K)}\left(\mathbb{C}\right)$$

and is as compatible as wanted. This isomorphism can be proved using the **Dirac-dual Dirac method** at the level of equivariant Kasparov's bivariant K-theory. Since periodic cyclic homology admits a close relationship with K-theory, our strategy for the right-hand side of (3) will rely on the adaptation of the Dirac-dual Dirac method from the Kasparov's bivariant K-theory to our needs, in order to recover this degree shift isomorphism at the level of periodic cyclic homology.

Aim of the talk: Adapt Dirac-dual Dirac method to periodic cyclic homology to show a homological analogue to the Baum-Connes conjecture.

The Dirac-dual Dirac method

Introduced by G. Kasparov, the bivariant K-theory assigns to a pair of $C^\star$-algebras (A, B) an abelian group $KK(A, B)$, generalizing both K-homology on the left variable and K-theory on the right one. It is endowed with the Kasparov product

$$- \times - : KK(A,B) \times KK(B,C) \longrightarrow KK(A,C). \tag{4}$$

Kasparov also developed an equivariant version of this theory preserving the product. He constructed two distinguished classes

$$\alpha \in KK_G(\mathcal{C}_\tau(G/K), \mathbb{C}) \ \text{ and } \ \beta \in KK_G(\mathbb{C}, \mathcal{C}_\tau(G/K)),$$

called *Dirac element* and *dual Dirac element*, where $\mathcal{C}_\tau(G/K) = \Gamma_0(G/K, \text{Cliff}(T^\star G/K))$ is the Clifford module of G/K. They satisfy the fundamental identities [Kas88]:

$$\alpha \times \beta = 1 \in KK_G\left(\mathcal{C}_\tau(G/K), \mathcal{C}_\tau(G/K)\right) \ \text{ and } \ \text{Res}_K^G(\beta \times \alpha) = 1 \in KK_K(\mathbb{C}, \mathbb{C}) = \text{Rep}(K).$$

These relations make α and β as dual to each others in a K-equivariant (but not G-equivariant!) way, and make the product by the Dirac element an isomorphism at the level of K-theory which is compatible with tensorization and crossed products. We will construct some analogues of these Dirac and dual Dirac elements at the level periodic cyclic homology which satisfy similar relations. To obtain these, we will need a more *algebraic* description of the bivariant K-theory, provided by the Cuntz-picture.

The Cuntz-picture

Since both K-theory and K-homology are generated by difference of classes, Cuntz's insight is that these functors can be *represented* by an algebra where difference of $\star$-homomorphisms become genuine algebraic elements. Given a $C^\star$-algebra A, the two-side ideal of differences qA of its $C^\star$-free product plays exactly this role. Cuntz approach then identifies bivariant K-theory with homotopy classes of $C^\star$-algebra homomorphisms between the Cuntz algebra qA and the algebra B stabilized by compact operators [Cun87]:

$$KK(A,B) \xrightarrow{\sim} \text{Hom}(qA, B \otimes_{C^\star} \mathcal{K})_{/\text{homotopy}}. \tag{5}$$

The Cuntz algebra verifies the fundamental property that there exists a natural and canonical continuous homotopy $qA \xrightarrow{\sim} A$, after stabilization via 2×2-matrices. Due to this property, the Kasparov product corresponds essentially in this setting to the composition of associated $\star$-homomorphisms. The Dirac and dual Dirac elements can be realized as G-equivariant $\star$-homomorphisms

$$\alpha : q(\mathcal{C}_\tau(G/K)) \longrightarrow \mathcal{K} \ \text{ and } \ \beta : q\mathbb{C} \longrightarrow M_2(\mathcal{C}_\tau(G/K)) \tag{6}$$

that are K-equivariant inverses to each others by the Kasparov theorem.

Results

In this talk we present an adaptation of the right-hand side of the Cuntz identification to fit the Banach/Fréchet context and produce a Fréchet counterpart of Dirac and dual Dirac elements. The idea to use the Cuntz-picture of bivariant K-theory to obtain a bivariant Chern character is due to Nistor [Nis93a] [Nis91]. We use here a slightly different approach, making heavily use of excision [CQ95], which was not yet known in the early nineties.

Our first idea is to propose a rescaling model for the Cuntz algebra qA. For any $R > 0$, we introduce a parametrized version of a free product, denoted Q_RA, as a completion of the algebraic free product by a rescaled norm. The associated ideal q_RA becomes the main receptacle of differences of Fréchet algebra homomorphisms out of A, of norm smaller than R. As before, we show that there exists a canonical and natural homotopy equivalence $q_R A \xrightarrow{\sim} A$, after stabilization by 2×2-matrices. It is compatible with tensorization and smooth crossed product constructions:

$$\widehat{CC}\big((q_R A \otimes_\pi A') \rtimes G\big) \xrightarrow{\sim} \widehat{CC}\big((A \otimes_\pi A') \rtimes G\big).$$

The second idea refines the $C^\star$-algebra $\mathcal{C}_\tau(G/K)$ into a Fréchet algebra $\mathcal{C}_\tau^\infty(G/K)$, constituted of its smooth vectors. The reason we consider this algebra is because the inclusion of the algebra $\mathcal{C}_c^\infty(G/K)$ of compactly-supported smooth functions into $\mathcal{C}_\tau^\infty(G/K)$ induces chain-homotopy equivalence compatible with tensorization and smooth crossed products:

$$\widehat{CC}\big(\mathcal{C}_c^\infty(G/K, A') \rtimes G\big) \xrightarrow{\sim} \widehat{CC}\big((\mathcal{C}_\tau^\infty(G/K) \otimes_\pi A') \rtimes G\big).$$

Proposition *There exists a parameter $R > 0$ great enough so that the Dirac and dual Dirac elements induce Fréchet algebra homomorphisms:*

$$\alpha^\sharp : q_R(\mathcal{C}_\tau^\infty(G/K)) \longrightarrow \mathcal{D} \quad \textit{and} \quad \beta^\sharp : q_1\mathbb{C} \longrightarrow M_2(\mathcal{C}_\tau^\infty(G/K)), \tag{7}$$

where $\mathcal{D}$ is a G-stable Fréchet subspace of the compact operators $\mathcal{K}$ with the absorbing property $\mathcal{D} \otimes_\pi \mathcal{D} \simeq \mathcal{D}$.

The previous proposition stands as a Fréchet counterpart of the Dirac and dual Dirac homomorphisms obtained in (6).

Theorem (*G., 2026*) *If $\mathcal{A} = A \otimes_\pi \mathcal{D}$ denotes the stabilization of A, the Dirac and dual Dirac homomorphisms descend as chain-complex homomorphisms*

$$(\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes G : \widehat{CC}(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \longrightarrow \widehat{CC}(\mathcal{A} \rtimes G)[\dim(G/K)],$$

$$(\beta^\sharp \otimes id_{\mathcal{A}}) \rtimes G : \widehat{CC}(\mathcal{A} \rtimes G) \longrightarrow \widehat{CC}(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G)[\dim(G/K)].$$

The homological analogue of the Baum-Connes conjecture with coefficients for Lie groups (3) relies on the fact that the morphisms above are homotopy inverse to each others. The proof of this statement follows two steps. First of all, the fundamental Kasparov identities (7) show that α and β are dual to each others in a K-equivariant way and the Cuntz-picture (5) relates this Kasparov product to the composition of the associated K-crossed products homomorphisms, which implies that

$$(\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes K : HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes K) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes K)$$

is an isomorphism of inverse $(\beta^\sharp \otimes id_{\mathcal{A}}) \rtimes K$. Secondly, the main theorem of Nistor (2), which identifies the periodic cyclic homology of crossed products by G and K around each conjugacy class of G, implies that $(\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes G$ is a quasi-isomorphism *on each stalk*. Since it is globally-defined, it induces a global isomorphism

$$(\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes G : HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes G),$$

which shows the main result.

Theorem (*G., 2026*) *Let G be a real Lie group with maximal compact subgroup K. For every complete locally convex algebra A endowed with a smooth action of G, the Dirac element $\alpha \in KK_G(\mathcal{C}_\tau(G/K), \mathbb{C})$ realizes the following isomorphism:*

$$(\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes G : HP_\bullet(\mathcal{A} \rtimes K) \simeq HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet + \dim(G/K)}(\mathcal{A} \rtimes G).$$

where $\mathcal{A} = A \otimes_\pi \mathcal{D}$ is the stabilization of A.

References

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